Dual Reciprocity Boundary Element Method for studying thermal flow in cooling Magna Oceans

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1. Overview

Earths early history is marked by a giant impact with a Mars-sized object which led to the formation of the moon. This impact event was the source of a substantial amount of melting of the Earths interior. Subsequent cooling of the Earth involved extensive crystallization in this "magma ocean" over a relatively short period of time. While chemical evidence from ancient sources provides some clues on the rate of cooling, computational models of such phenomena are sparse.

The presented work uses the dual reciprocity boundary element method (DRBEM) to model heat flow in a multiphase fluid. DRBEM extends on the boundary element method (BEM) allowing one to solve the heat equation only on the boundary of the problem, avoiding expensive rediscretization found in traditional methods. DRBEM works by approximating the residual term of the PDE that would be troublesome to use in BEM by a linear combination of radial basis functions chosen *a priori*. Using the approximation of the residual portion allows for the boundary method to be applied to more complicated PDEs such as the heat equation. The research presented extends on DRBEM to solve the heat equation in an infinite magma ocean with multiple advecting crystals.

2. Governing equations and boundary integrals

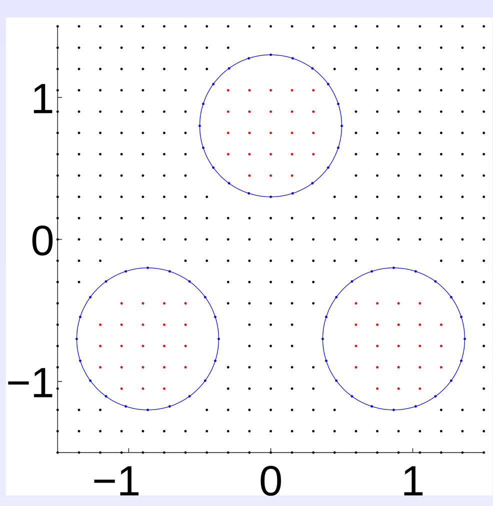


Figure 1: The mesh for the three crystal problem. The boundary elements are approximated using cubic spline interpolation. Even though there are domain nodes, they are not structured and are not encumbering during rediscretization.

Modeling the crystal settling behavior requires solving a coupled system of partial differential equations. To facilitate using DRBEM, a system of equations is written for the $P+1$ domains. The bounded domains $\{\Omega^p\}_{p=1}^P$ represent the P crystals in the suspension. The unbounded domain representing the infinite magma ocean is denoted Ω_0.

$$-\nabla p + \mu^p \nabla^2 \boldsymbol{u} + \rho^p \boldsymbol{b} = 0 \quad \boldsymbol{x} \in \Omega^p \quad p = \{0, 1, \dots, P\} \tag{1}$$

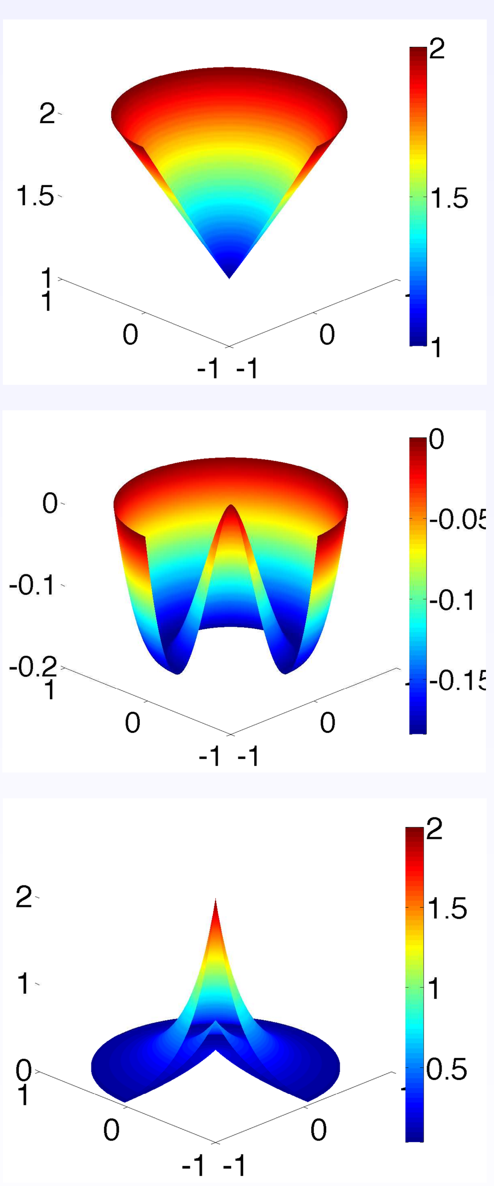
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta - \kappa^p \nabla^2 \theta = b \quad \boldsymbol{x} \in \Omega^p \quad p = \{0, 1, \dots, P\} \tag{2}$$

where μ^p and ρ^p are the viscosity and density of the fluid in domain Ω^p, and κ^p is the thermal diffusivity of the p-th domain.

Applying standard BEM and DRBEM procedures, (1) and (2) can be rewritten as a system of boundary integral equations [1][2]:

$$\boldsymbol{u}_i = \boldsymbol{u}_\infty + \frac{2}{1+\lambda_i}\left[-\frac{1}{4\pi\mu^i}\sum_{p=1}^{P}\int_{\Gamma^p} \Delta \boldsymbol{f}^p \cdot \mathbf{J}^i \, \mathrm{d}\Gamma^p + \sum_{p=1}^{P}\frac{1}{2\pi}(1-\lambda^p)\int_{\Gamma^p} \hat{\boldsymbol{n}} \cdot \mathbf{K}^i \cdot \boldsymbol{u} \, \mathrm{d}\Gamma^p\right] \tag{3}$$

$$\theta_i = \sum_{p=0}^{P}\frac{1}{\kappa_p}\sum_{j=1}^{J_p}\beta_j^p\left(c_i^{p,j}\hat{\theta}_i^{p,j} + \int_{\Gamma^p}\left[\left(\nabla\hat{\theta}^{p,j}\cdot\hat{\boldsymbol{n}}^{\pm}\right)\theta^{*i} - \left(\nabla\theta^{*i}\cdot\hat{\boldsymbol{n}}^{\pm}\right)\hat{\theta}^{p,j}\right]\mathrm{d}\Gamma^p\right) \tag{4}$$



where $\mathbf{J}$ and $\mathbf{K}$ are Greens functions, $\Delta \boldsymbol{f}$ the jump in surface tension, and $\lambda^p = \mu^p/\mu^0$ is the viscosity ratio for the Stokes equation. For the heat equation θ^* is the Greens function for the Laplace operator and $\hat{\theta}^{p,j}$ is the the solution to

$$\nabla^2 \hat{\theta}^{p,j} = f^{p,j}, \tag{5}$$

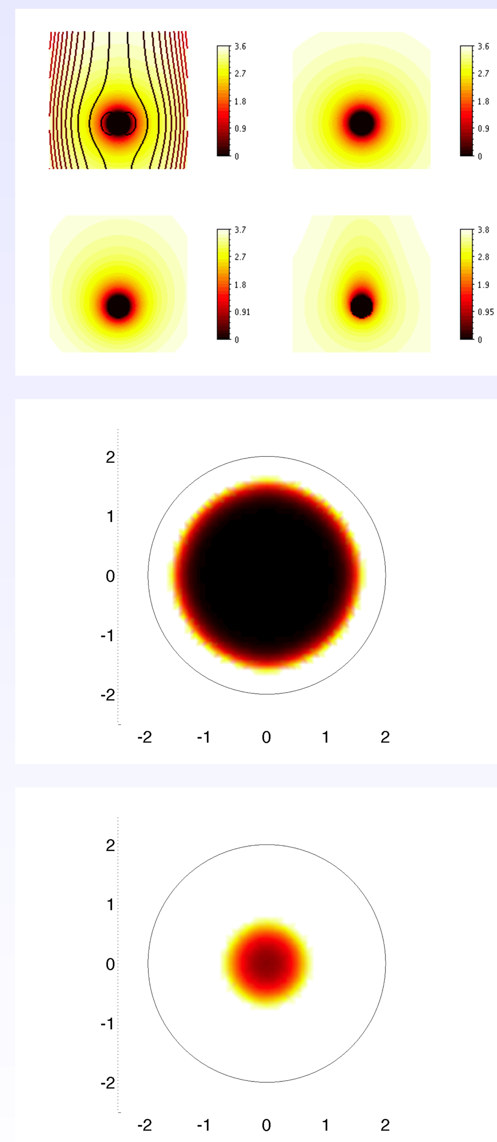
with β_j^p chosen such that $\sum_j \beta_j^p f^{p,j} \approx \frac{\partial\theta}{\partial t} + \vec{u}\cdot\nabla\theta - b$ in Ω_p, where $f^{p,j}$ are appropriate radial basis functions (RBFs). Numerical methods for solving (3) have been presented [3][4] and (4) [5][6] on bounded domains. The current work focuses on numerical solutions to (4) for the multicrystal system.

There are several choices for RBFs to use to approximate the residual term in DRBEM. Historically the cone has been used due to its simplicity. However, research has shown that thin-plate splines (TPS) perform better numerically and have preferable mathematical properties [7].

The asymptotic behavior of the cone and TPS RBFs preclude them from being used in unbounded domains. Loeffler and Mansur [8] developed a RBF that satisfies the necessary requirements. The LM RBFs require a parameter c to be chosen *a priori*.

Figure 2: (Top) Cone RBF $(1+r)$. (Middle) TPS RBF $(r^2 \ln r)$. (Bottom) LM RBF $(2c-r)/(r+c)^4$ with, from top to bottom, $c = 1, 1.5, 2$. With smaller values of c the LM RBF only approximate local behavior. However, as c becomes large, the linear system needed to find the RBF coefficients becomes ill conditioned.

3. Numerical Results



A closed form asymptotic solution exists for limited cases of thermal flow around a single particle. The analytical solutions allows for the heat and fluid velocities to be calculated without the need for approximation or large scale computation. However, the solution only hold in limited cases. The solution is only valid for one near spherical crystal in a fluid with a Péclet number close to 1. While solutions of this type can offer intuition into the behavior of larger problems, they cannot rigorously provide results.

The numerical method being developed based on the modified DRBEM method is able to approximate the heat and fluid velocity in a problem with multiple settling crystals. It also is able to provide a higher order approximation of the solution in the case the crystals generic in shape and the Péclet is arbitrary. In return DRBEM requires assembling and solving a large linear system.

Figure 3: Analytics result for one crystal (top). Numerical result for one crystals at $t = 50$ (middle), and one crystal at $t = 250$ (bottom), with crystal thermal diffusivity set at 0.5 and the thermal diffusivity of the magma ocean set at 1.0. The numerical results show the diffusion of the heat with initial condition $\theta = 1$ in the suspension and $\theta = 0$ in the particles. The crystal eventually melts as it is in an infinite magma ocean. However, it is able to temporarily cool the surrounding region.

4. Discussion

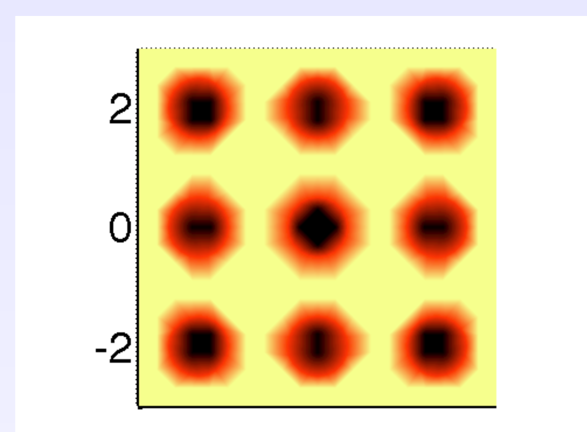


Figure 4: The initial conditions of the for the nine particle problem. The implementation of the modified DRBEM will be robust enough to handle an arbitrary number of particles with arbitrary shapes and physical properties.

Already multiphase fluid flow can be accurately and efficiently computed using traditional BEM. Using a boundary method allows for the simulation of many viscous crystals advecting through an infinite suspension fluid. By using DRBEM, this ability is extended to the heat equation. The present research will allow for the simulation of a hydrothermal system with hundreds of settling crystals advecting and thermally interacting with the surrounding magma ocean.

DRBEM allows for a meshless approach to solving a system of partial differential equations. The meshless feature of the method enables the interface between the crystals and the magma ocean to be more accurately approximated. In addition, any rediscretization required only occurs along the boundary of the crystals.

In the near future, the implemention of the modified DRBEM is method will have the ability to model the settling of hundreds of cool crystals into a hot magma ocean. The simulations will provide an efficient way to measure cooling in an environment not unlike the magma ocean found in the early Earth after the large impact event. Data collected may be able to bring tighter bounds on rates of cooling of the planet.

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