Dual Reciprocity Boundary Element Method for studying thermal flow in cooling Magna Oceans (DI23A-2071)

Tyler Drombosky† and Saswata Hier-Majumder‡*

† Applied Mathematics and Scientific Computing, University of Maryland: College Park, drombosk@math.umd.edu

‡ Department of Geology, University of Maryland: College Park, saswata@umd.edu

* Center for Scientific Computation and Mathematical Modeling, University of Maryland: College Park

1. Overview

Earth's early history is marked by a giant impact with a Mars-sized object which led to the formation of the moon. This impact event was the source of a substantial amount of melting of the Earth's interior. Subsequent cooling of the Earth involved extensive crystallization in this "magma ocean" over a relatively short period of time. While chemical evidence from ancient sources provides some clues on the rate of cooling, computational models of such phenomena are sparse.

The presented work uses the dual reciprocity boundary element method (DRBEM) to model heat flow in a multiphase fluid. DRBEM extends on the boundary element method (BEM) allowing one to solve the heat equation only on the boundary of the problem, avoiding expensive rediscretization found in traditional methods. DRBEM works by approximating the residual term of the PDE, which would be troublesome to use in BEM, by a linear combination of radial basis functions chosen *a priori*. Using the approximation of the residual allows for the boundary method to be applied to more complicated PDEs such as the heat equation. The research presented extends on DRBEM to solve the heat equation in a bounded magma ocean with multiple advecting crystals.

2. Governing equations and boundary integrals

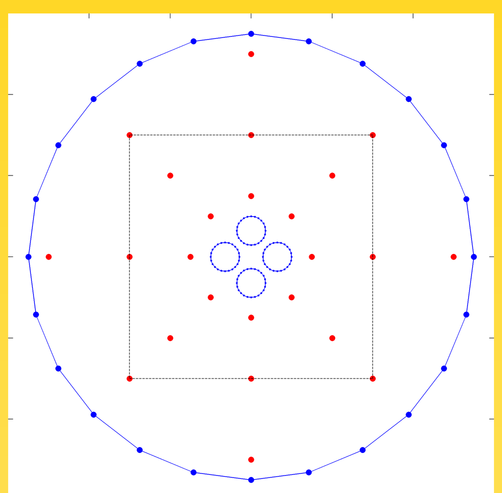


Figure 1: The mesh for the four crystal problem. The boundary elements are approximated using cubic spline interpolation. The domain nodes are not structured and thus are not encumbering during rediscretization.

Modeling the crystal settling behavior requires solving a coupled system of partial differential equations. To facilitate using DRBEM, a system of equations is written for the $P+1$ domains. The bounded domains $\{\Omega^p\}_{p=1}^{P}$ represent the P crystals in the suspension. The bounded domain representing the infinite magma ocean is denoted Ω_0.

$$-\nabla p + \mu^p \nabla^2 \boldsymbol{u} + \rho^p \boldsymbol{b} = 0 \quad \boldsymbol{x} \in \Omega^p \quad p = \{0, 1, \ldots, P\} \qquad (1)$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta - \kappa^p \nabla^2 \theta = b \quad \boldsymbol{x} \in \Omega^p \quad p = \{0, 1, \ldots, P\} \qquad (2)$$

where μ^p and ρ^p are the viscosity and density of the fluid in domain Ω^p, and κ^p is the thermal diffusivity of the p-th domain.

Applying standard BEM and DRBEM procedures, (1) and (2) can be rewritten as a system of boundary integral equations [1][2]:

$$\boldsymbol{u}_i = \boldsymbol{u}_\infty + \frac{2}{1+\lambda_i}\left[-\frac{1}{4\pi\mu^i}\sum_{p=1}^{P}\int_{\Gamma^p} \Delta \boldsymbol{f}^p \cdot \mathbf{J}^i \, \mathrm{d}\Gamma^p + \sum_{p=1}^{P} \frac{1}{2\pi}(1-\lambda^p)\int_{\Gamma^p} \hat{\boldsymbol{n}} \cdot \mathbf{K}^i \cdot \boldsymbol{u} \, \mathrm{d}\Gamma^p\right] \qquad (3)$$

$$\theta_i c_i^p + \int_{\Gamma^p} \left[(\nabla\theta \cdot \hat{\boldsymbol{n}})\,\theta^{*i} - \left(\nabla\theta^{*i} \cdot \hat{\boldsymbol{n}}\right)\theta\right] \mathrm{d}\Gamma^p = \frac{1}{\kappa_p}\sum_{j=1}^{J_p} \beta_j^p \left(c_i^{p,j}\hat{\theta}_i^{p,j} + \int_{\Gamma^p}\left[\left(\nabla\hat{\theta}^{p,j} \cdot \hat{\boldsymbol{n}}\right)\theta^{*i} - \left(\nabla\theta^{*i} \cdot \hat{\boldsymbol{n}}\right)\hat{\theta}^{p,j}\right] \mathrm{d}\Gamma^p\right) \qquad (4)$$

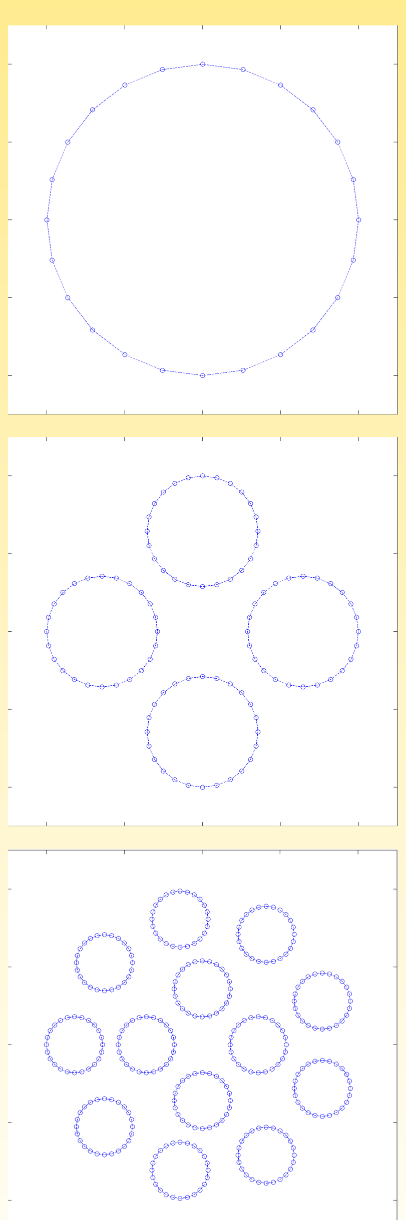


Figure 2: (Top) One crystal. (Middle) Four crystals. (Bottom) Thirteen crystals. Each particle has 24 boundary nodes. The crystals are configured to fit into a larger circle for a given radius and minimum separation.

where $\mathbf{J}$ and $\mathbf{K}$ are Greens functions, $\Delta \boldsymbol{f}$ the jump in surface tension, and $\lambda^p = \mu^p/\mu^0$ is the viscosity ratio for the Stokes equation. For the heat equation θ^* is the Greens function for the Laplace operator and $\hat{\theta}^{p,j}$ is the solution to

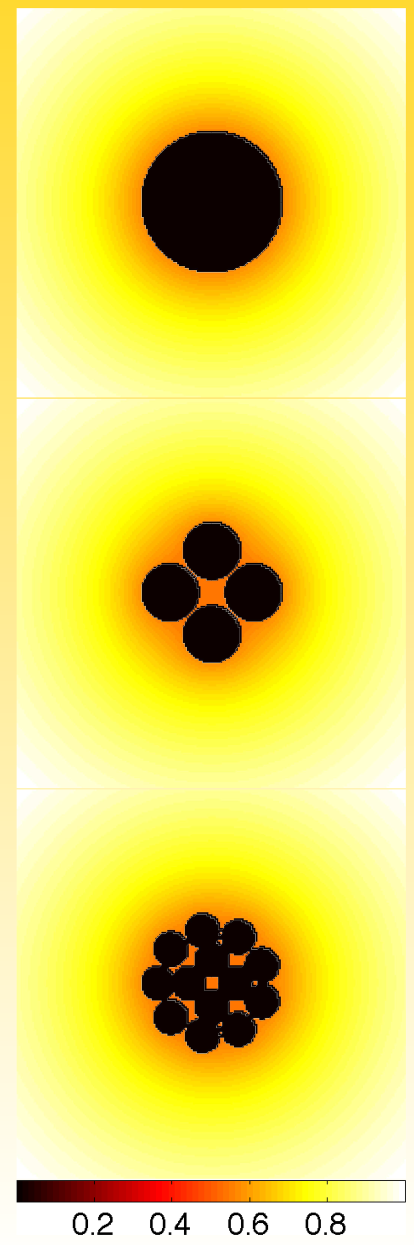
$$\nabla^2 \hat{\theta}^{p,j} = f^{p,j}, \qquad (5)$$

with β_j^p chosen such that $\sum_j \beta_j^p f^{p,j} \approx \frac{\partial\theta}{\partial t} + \vec{u} \cdot \nabla\theta - b$ in Ω_p, where $f^{p,j}$ are appropriate radial basis functions (RBFs). Numerical methods for solving (3) have been presented [3][4] and (4) [5][6] on bounded domains. The current work focuses on numerical solutions to (4) for the multicrystal system.

The linear system for each domain is written separately. The potential and flux across the particle-suspension boundary are matched at the linear algebra stage. The crystals have constant temperature and the steady state temperature for the suspension in the tank is computed.

The ultimate goal is to solve the transient problem with thermal interactions between the particles and the suspension. However, the linear system produced by DRBEM is often highly ill-conditioned leading to instability in the marching schemes.

3. Numerical Results



The steady state thermal conduction for several crystal configurations is calculated numerically. DRBEM only solves for the solution on the boundary of the problem, however the solution on the boundary can be used to solve for solution in the domain. This can be done to obtain high-resolution images of the thermal diffusion or gather profile samples.

From the numerical simulations, one can see that the thermal profile grows as more crystals are added to the configuration. The fact that the steady state thermal profile are as such indicates that a cluster of several crystals cooling in tandem have a greater cooling affect than isolated particles. This results show that it may be possible to effective cool a large amount of fluid with smaller arrays of cool crystals.

In the near future, the model will be coupled with a Stokes flow solver to allow for the thermal advection of the system to be computed. By using BEM and DRBEM the solution only needs to be computed on the boundary. Using fast methods allows for the thermal profiles to be computed quickly and efficiently.

Figure 3: Numerical solution for one crystal (top). Numerical solution for four crystals (middle). Numerical solution for seventeen crystals (bottom). The numerical results show the steady state heat diffusion of the system where the temperature at the crystal boundaries is constant.

4. Discussion

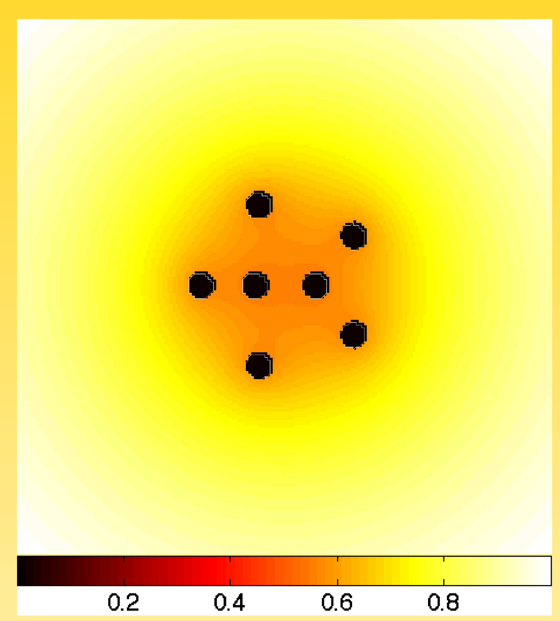


Figure 4: The model and solver can handle arbitrary crystal configurations along with steady state velocities and heat sources. This provides the tools necessary to compare thermal properties across various configurations.

Heat advection can be accurately modeled in large micro-scale environments, preserving the interaction between discrete grains. The results can also be applied to homogenization problems for more accurate macro-scale simulations.

Already multiphase fluid flow can be accurately and efficiently computed using traditional BEM. Using a boundary method allows for the simulation of many viscous crystals advecting through an infinite suspension fluid. By using DRBEM, this ability is extended to the heat equation. The present research will allow for the simulation of a hydrothermal system with hundreds of settling crystals advecting and thermally interacting with the surrounding magma ocean.

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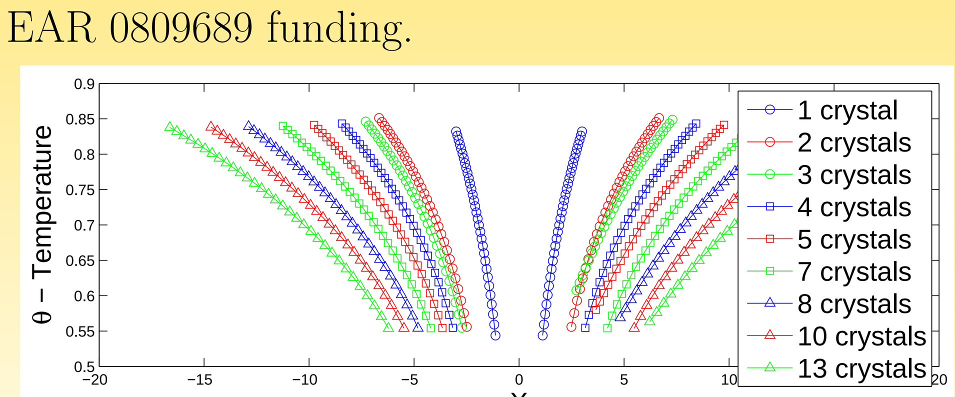

Figure 5: The thermal profiles for various crystal configurations. When several crystals of the same size are grouped together, the far field cooling effect is higher compared to one crystal cooling.

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